Gaps in the lattices of topological group topologies

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3. successors of locally compact abelian group (G, τ) in $\mathcal{PG}(G)$ and $\mathcal{G}(G)$

The study of lattices of topologies over a set was initiated by Birkhoff around 1930s. From then on, there are a lot of study in this direction. Let $\mathcal{L}(X) = (\mathcal{T}(X), \wedge, \vee)$ be the lattice of all topologies on a set X, where the binary operations $\vee$ and $\wedge$ are called the *join* and *meet*, respectively. As usual, the join $\tau \vee \sigma$ of topologies $\tau, \sigma \in \mathcal{T}(X)$ is the coarsest topology λ on X satisfying $\tau \subset \lambda$ and $\sigma \subset \lambda$. Similarly, $\tau \wedge \sigma$ is the finest topology λ^* on X satisfying $\lambda^* \subset \tau$ and $\lambda^* \subset \sigma$. It is known and easy to verify that the lattice $(\mathcal{T}(X), \wedge, \vee)$ is *complete*, and family $\mathcal{L}_1(X)$ of T_1 topologies on X forms a *sublattice* of $\mathcal{T}(X)$.

Theorem 1 (Birkhoff,1935)

For every set X, $\mathcal{L}(X)(\mathcal{L}_1(X))$ *is a complete lattice.*

Our main concern is to study the lattices of (para)topological group topologies on a group G. From now on, we will fix
$X = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a > 0, a, b \in \mathbb{R}\} \leq GL(2, \mathbb{R})$.
Let $(\mathcal{PG}(G), \wedge, \vee)$ be the lattice of all paratopological group topologies on a group G, where the binary operations $\vee$ and $\wedge$ are called the *join* and *meet*, respectively. As usual, the join $\tau \vee \sigma$ of topologies $\tau, \sigma \in \mathcal{PG}(G)$ is the coarsest paratopological group topology λ on G satisfying $\tau \subset \lambda$ and $\sigma \subset \lambda$. Similarly, $\tau \wedge \sigma$ is the finest paratopological group topology λ^* on G satisfying $\lambda^* \subset \tau$ and $\lambda^* \subset \sigma$. It is known and easy to verify that the lattice $(\mathcal{PG}(G), \wedge, \vee)$ is *complete*, and family $\mathcal{G}(G)$ of topological group topologies on G forms a *sublattice* of $\mathcal{PG}(G)$.

Definition 2

Let G be an abstract group and $\mathcal{S}$ be a subfamily of the lattice $\mathcal{PG}(G)$ of all paratopological group topologies on G. A pair of elements $\tau, \sigma \in \mathcal{S}$ with $\sigma \subsetneq \tau$ is a *gap* in $\mathcal{S}$ if no element $\lambda \in \mathcal{S}$ satisfies $\sigma \subsetneq \lambda \subsetneq \tau$. If $\{\sigma, \tau\}$ is a gap in $\mathcal{S}$, and $\sigma \subseteq \tau$, then τ is called a *successor* of σ in $\mathcal{S}$ and σ is a *predecessor* of τ in $\mathcal{S}$.

Let $\mathcal{T}$ be a nondiscrete (para)topological group topology on the additive group of integers, $\mathbb{Z}$. By the Kuratowski–Zorn lemma, $\mathcal{T}$ is contained in a *maximal* (by inclusion) nondiscrete (para)topological group topology on $\mathbb{Z}$, say, $\mathcal{T}^*$. In what follows the term *maximal* topology will always refer to a non-discrete topology. Therefore $\{\mathcal{T}^*, \tau_d\}$ is a *gap* in $\mathcal{PG}(\mathbb{Z})$.

We now use maximal topologies to define predecessors of τ_u in $\mathcal{G}(\mathbb{R})$ and $\mathcal{PG}(\mathbb{R})$.

Example 3 (He–Peng–Tkachenko–Xiao, 2019)

Let $\mathcal{T}^*$ be a maximal (para)topological group topology on $\mathbb{Z}$ and $\mathcal{T}^*(0)$ be the family of all sets $U \in \mathcal{T}^*$ with $0 \in U$. Then the family

$$\mathcal{B} = \{U + (-\varepsilon, \varepsilon) : U \in \mathcal{T}^*(0),\ \varepsilon > 0\}$$

is a local base at zero for a (para)topological group topology τ on $\mathbb{R}$ and τ is a predecessor of τ_u in $\mathcal{G}(\mathbb{R})$ (respectively, in $\mathcal{PG}(\mathbb{R})$).

Theorem 4 (He–Peng–Tkachenko–Xiao, 2019)

Let σ be a predecessor of τ_u in $\mathcal{G}(\mathbb{R})$ (in $\mathcal{PG}(\mathbb{R})$). Then $\sigma\restriction\mathbb{Z}$ is a maximal (para)topological group topology on $\mathbb{Z}$.

Combining Example 3 and Theorem 4 we obtain a complete description of the predecessors of τ_u in the lattices $\mathcal{G}(\mathbb{R})$ and $\mathcal{PG}(\mathbb{R})$. In fact, the operations on topologies described in Example 3 and Theorem 4 are mutually inverse. In other words, if σ is a predecessor of τ_u in $\mathcal{G}(\mathbb{R})$ or $\mathcal{PG}(\mathbb{R})$ and $\mathcal{T} = \sigma\restriction\mathbb{Z}$, then $\mathcal{T}$ is a maximal (para)topological group topology on $\mathbb{Z}$ and the family $\{U + (-\varepsilon, \varepsilon) : 0 \in U \in \mathcal{T},\ \varepsilon > 0\}$ is a local base at zero for the topology σ.

Let us show that all predecessors of the topology τ_u 'inherit' the Hausdorff separation property, independently of whether they are taken in $\mathcal{G}(\mathbb{R})$ or in $\mathcal{PG}(\mathbb{R})$.

Proposition 1 (He–Peng–Tkachenko–Xiao, 2019)

If σ is a predecessor of τ_u either in $\mathcal{G}(\mathbb{R})$ or $\mathcal{PG}(\mathbb{R})$, then the topology σ is Hausdorff.

Theorem 5

Let (G, τ) be a Hausdorff topological abelian group. Then all predecessors of τ in $\mathcal{G}(G)$ (if exist) are Hausdorff if and only if the group G is torsion free.

Let us recall that a Hausdorff topological group G is *minimal* if it does not admit a strictly coarser Hausdorff topological group topology. Theorem 5 implies the following curious fact about minimal topological abelian groups.

Corollary 6

Let (G, τ) be a minimal topological abelian group. If G is torsion free, then τ has no predecessors in $\mathcal{G}(G)$.

In [4], we have proved that a minimal abelian torsion-free group G have no predecessors in $\mathcal{G}(G)$. But we obtain the opposite result for the group X.

Example 7

Let $\mathcal{U}$ be the family of subsets of X of the form

$$W_n = \{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in X : a \in (1 - 1/n, 1 + 1/n),\ b \in \mathbb{R}, \},$$

where $n \in \mathbb{N}^+$. Then there exists a topological group topology $\mathcal{T}_{\mathcal{U}}$ on X with local base $\mathcal{U}$ at the identity I of X. The topology $\mathcal{T}_{\mathcal{U}}$ is strictly coarser than the Euclidean topology of the group X and $\mathcal{T}_{\mathcal{U}}$ is the unique predecessor of τ_X in $\mathcal{G}(X)$.

Problem 8

Does every locally compact noncompact minimal Hausdorff group admit a strictly weaker (minimal) Hausdorff paratopological group topology?

Remark 1

In fact, these groups are locally compact noncompact minimal group when endowed with the discrete topology. Then the natural question as to whether every infinite non-topologizable group admits a non-discrete Hausdorff (minimal) paratopological group topology is a special case of Problem 8.

Problem 9

Does every infinite non-topologizable group admit a non-discrete Hausdorff (minimal) paratopological group topology?

Proposition 2 (He–Peng–Tkachenko–Xiao, 2019)

Let $\{\tau_1, \tau_2\}$ be a gap in the lattice of all (para)topological group topologies on a group G. If H is a central group of G, then either $\tau_1\restriction H = \tau_2\restriction H$ or $\{\tau_1\restriction H, \tau_2\restriction H\}$ is a gap in the lattice of (para)topological group topologies on H. Further, if $q\colon G \to K$ is a surjective homomorphism of groups and $q(\tau_i) = \{q(U) : U \in \tau_i\}$ for $i = 1, 2$, then $\{q(\tau_1), q(\tau_2)\}$ is a gap in the lattice of (para)topological group topologies on K provided that $q(\tau_1) \neq q(\tau_2)$.

Proposition 3

The gaps in the lattice of topological group topologies over the group X is not preserved by taking normal subgroups.

Theorem 10

Let N be a complete subgroup of a Hausdorff topological abelian group G with topology τ such that the quotient group G/N is compact. Then there exists a one-to-one correspondence between the predecessors, $\mathcal{P}_2(\tau)$, of τ in $\mathcal{G}_2(G)$ and the predecessors, $\mathcal{P}_2(\tau\upharpoonright N)$, of $\tau\upharpoonright N$ in $\mathcal{G}_2(N)$. This correspondence is the restriction mapping $\sigma \mapsto \sigma\upharpoonright N$, where $\sigma \in \mathcal{P}_2(\tau)$.

Let $G = \mathbb{R}$ and $N = \mathbb{Z}$. The cardinality of the family $\mathcal{P}_2(\tau)$ of predecessors of τ_u in $\mathcal{G}_2(\mathbb{R})$ is equal to the cardinality of the family of all maximal topological group topologies on $\mathbb{Z}$. The latter number is 2^c. Since every predecessor of τ_u in $\mathcal{G}(\mathbb{R})$ is a Hausdorff topology, we have the following result.

Corollary 11

The usual interval topology τ_u on the additive group of reals has exactly 2^c predecessors in the lattice $\mathcal{G}(\mathbb{R})$.

Theorem 12 (He–Peng–Tkachenko–Xiao, 2019)

A compact Hausdorff topological group topology τ on a divisible abelian group G has no successors in $\mathcal{G}(G)$.

Corollary 13

For any positive integer n, the usual Euclidean topology of R^n does not have successors in $\mathcal{G}(R^n)$.

Theorem 14

The Euclidean topology τ_X on X has no successors in $\mathcal{G}(X)$.

Theorem 15

Let (G, τ) be a connected LCA group. Then τ has no successors in $\mathcal{G}(G)$.

Proposition 4 (He–Peng–Tkachenko–Xiao, 2019)

For every integer $n \geq 0$, the pair $\{\tau_u^{n+1}, \tau_u^n \times \tau_s\}$ is a gap in $\mathcal{PG}(\mathbb{R}^{n+1})$, where τ_u^k is the usual Euclidean topology on $\mathbb{R}^k$ for $k \in \{n, n+1\}$ and $\tau_u^n \times \tau_s$ is the topology of $(\mathbb{R}^n, \tau_u^n) \times (\mathbb{R}, \tau_s)$.

Theorem 16 (He–Peng–Tkachenko–Xiao, 2019)

Let $X = \{x_\alpha : \alpha < \mathfrak{c}\}$ be a Hamel base for $\mathbb{R}$ over the field $\mathbb{Q}$, where $x_0 = 1$. Let X be the disjoint union of its proper subsets X_0 and X_1 and we always assume that $x_0 \in X_0$. Then

$$M_i = \Big\{ \sum_{j=1}^{n} q_j x_{\alpha_j} : q_j \in \mathbb{Q},\ x_{\alpha_j} \in X_i \text{ for each } j = 1, \ldots, n \Big\}$$

is a dense subgroup of $(\mathbb{R}, \tau_u)$ for $i = 0, 1$ and $\mathbb{R} = M_0 \oplus M_1$.
For every $q \in M_0$, let

$$U_q = \{p + m : p > q,\ p \in M_0,\ m \in M_1\}.$$

Then the family

$$\mathcal{F} = \{(-1/n, 1/n) \cap U_{-1/n} : n \in \mathbb{N}^+\}$$

is a local base at zero for a paratopological group topology σ on $\mathbb{R}$ which is a successor of τ_u in the lattice $\mathcal{PG}(\mathbb{R})$.

Suppose $\{\tau_u, \sigma\}$ is a gap in $\mathcal{PG}(\mathbb{R})$ with countable character at 0 of $(\mathbb{R}, \sigma)$. We assume that $\{U_n : n \in \mathbb{N}\}$ is a neighbourhood base at 0 of $(\mathbb{R}, \sigma)$ with the following condition $U_n \subseteq (-1/n, 1/n)$ and $U_n + U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$. Hence the constructions given in Proposition 4 and Theorem 16 satisfy the above's conditions. Then the exponential map will induce a paratopological group topologies on $\mathbb{R}^+$ if $\mathbb{R}$ endowed with the paratopological group topology σ. And the pair $\{\exp(\tau_u), \exp(\sigma)\}$ also forms a gap in $\mathcal{PG}(\mathbb{R}^+)$ since exponential map is an isomorphism.

Example 17

Let X be the subgroup of $GL(2, \mathbb{R})$ which consists of all matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$, where $a > 0$ and $b \in \mathbb{R}$ is arbitrary. There exists a Hausdorff paratopological group topology τ on X whose base at the identity I of X is formed by the sets

$$W_n = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a \in \exp(U_n),\ |b| < 1/n \right\},$$

with $n \in \mathbb{N}^+$. And $\{\tau_X, \tau\}$ is a gap in the sup semilattice of Hausdorff paratopological group topologies on X.

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Thank you!